Escalating Games, or why Proportionate Responses Can Fail in Preventing Conflicts*

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Abstract

We argue that cooperation can become more fragile if (i) there are sufficiently many intermediate levels of cooperation and (ii) players cannot respond with large punishments to small deviations. A failure to credibly commit to disproportionate punishments can stem from legal or political feasibility, or from historical precedent. Specifically, we show that regardless of how patient the players are, any prisoner's dilemma game can be extended with intermediate levels of cooperation in such a way that full conflict is the only equilibrium outcome of the extended game in a class of strategies with limited punishment.

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1 Introduction

In his seminal book, "The Strategy of Conflict," Thomas Schelling spoke of the difficulty of committing to a big retaliation in response to a small provocation. He was suggesting that in the face of this difficulty the possibility to wage a limited war, i.e. the availability

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of intermediate actions, might prevent the conflict from escalating: "if it [the threat] can be decomposed into a series of consecutive smaller threats, there is an opportunity to demonstrate on the first few transgressions that the threat will be carried out on the rest." More specifically, in the context of a repeated prisoner's dilemma situation, Schelling suggested that the presence of intermediate actions might actually allow for deescalation of conflict (see pp. 45–46).

At the same time, many historical conflicts arguably have evolved through a series of escalating steps before they broke into large scale military engagements. Initial provocations were typically met with counter-measures, which, although considered to be proportionate at the time, failed to prevent the consequent escalation. In this paper, we argue that such escalations can arise if—contrary to Schelling's intuition—there is a sufficiently large number of intermediate steps, and if—contrary to folk political wisdom—punishments are restricted to being proportionate. Specifically, we show that in the absence of a credible commitment to a full-out retaliation, for any prisoner's dilemma game and any choice of discount factor cooperation can break down if there is a sufficient number of suitably chosen intermediate actions.

The underlying mechanism, that we study in detail further, can be demonstrated using the following prisoner's dilemma game as an example:

	C	D
C	$7,7$	$0,11$
D	$11,0$	$3,3$

The cooperative outcome (C, C) can be supported as an equilibrium in every round of a repeated game as long as $11 + 3\delta/(1-\delta) \leq 7/(1-\delta)$ or $\delta \geq 1/2$.

Now, suppose that an intermediate level of cooperation (or conflict) is available:

	C	D'	D
C	$7,7$	$1,10$	$0,11$
D'	$10,1$	$5,5$	$1,8$
D	$11,0$	$8,1$	$3,3$

Further, suppose that the players cannot punish deviations from C to D' by playing D. For example, and judging by the existing precedents, the Unites States is unlikely to respond with a large scale military engagement (D) following a single act of a cyber attack against the US (D'). Consequently, if a player considers a deviation to D', she can reasonably expect to be punished with the same action D' rather than with D. In this case, the deviation from (C, C) is profitable if $10 + 5\delta/(1-\delta) > 7/(1-\delta)$ or $\delta < 3/5$. If we suppose that (D', D') is played as a steady state, then a deviation to D is also profitable as long as $\delta < 3/5$. Thus, for any $\delta \in [1/2, 3/5)$ cooperation can be sustained as an equilibrium outcome in the smaller game (or in a game with strong punishments) but it is not sustainable in the extended game when punishments are limited.

We can generalize the preceding example by studying a prisoner's dilemma game with an arbitrary number of intermediate levels of deviation (or cooperation). We will further assume that grim strategies cannot be used credibly, i.e. a player cannot respond with a punishment that is stronger than the original deviation. Given our assumption on proportional punishments, we will show that irrespective of how patient the players are, there are intermediate levels of deviation such that no cooperative equilibria exist.

We view our primary assumptions as representative of the real international relations. Firstly, countries can indeed engage in cooperation or conflict on various levels: from foreign direct investments to financial sanctions, from coordinated development of global IT networks to cyber warfare, from technological cooperation to espionage, from joint military exercises to locating strategic military installations closer to their opponents. Secondly, we see that actions in the international arena mostly cause proportionate responses, and hence are expected to be of such nature. Examples abound: trade tariffs, expulsions of diplomats, military exercises, proxy wars. Countries are forced to respond to provocations to maintain the opponents' beliefs in their resolve, but they are careful not to escalate the situation. Yet, as we argue, this general approach might in itself lead to escalation.

Our assumptions not only fit those situations where open conflicts arise but they are also suitable to study the build-up of tensions prior to a potential conflict. In particular, in arm races there are multiple intermediate levels of armament and the players typically play symmetric response strategies in expanding their arsenals. What we argue then, is that even if no player were to consider it profitable to go from zero armament to, say, having nuclear capabilities in one go, the same players might end up with nuclear arsenals nevertheless if sufficiently many intermediate levels of armament are possible and if neither opponent can commit to a grim strategy.

We contribute to the literature that studies conflicts by highlighting a new dimension to the commitment problem, the one that only arises when intermediate levels of cooperation (or conflict) are considered. While multi-stage games of conflict have been considered in the earlier literature, see e.g. Powell (1987) or the setup in Bueno de Mesquita et al. (1997),[1] there are not many papers with an emphasis on the intermediate levels of conflict in connection with the proportionality of responses. The only paper that we know of and the one with a setup most similar to ours is McGinnis (1991), who models intermediate levels of cooperation as a sequence of overlapping prisoner's dilemma games. He shows that if the payoff function takes a specific log-linear form, then the equilibrium will likely be sustained at one of the intermediate levels of cooperation. In contrast with McGinnis, we study a general payoff structure and show that a more extreme outcome—namely, no cooperation—is always a possibility.

The literature on prisoner's dilemma with intermediate actions is also relatively scarce. Snidal (1985) shows that new strategic difficulties arise in such games, e.g., having multiple Pareto-efficient outcomes instead of just one. He does not, however, speak of the possibility of conflict escalation. In an independent work, Langlois (1989) explicitly allows for conflict escalation. He considers a repeated prisoner's dilemma game with a continuum of intermediate actions coupled with linear payoffs, and he shows that there exists a Markov equilibrium in linear strategies that can sustain full cooperation. In comparison with our work, Langlois does not impose any restrictions on the degree of punishment, whereas such restrictions are the main focus of our discussion.

In a later work, Friedman and Samuelson (1990) analyze repeated games with continuous payoffs, thus similar to Langlois (1989), but the authors restrict the punishment to be proportional to the deviation. If the deviation approaches zero, so does the punishment. Friedman and Samuelson consider reference dependent strategies, and show that that if the discount factor is large enough, then with these strategies it is possible to construct a deescalating equilibrium despite having limited punishment. In contrast, we consider the limiting behaviour of discrete games and simpler "tit-for-tat" style strategies. We arrive at the opposite conclusion: for every value of the discount factor there are games with sufficiently many intermediate actions where escalation cannot be precluded. The difference between our result and that of Friedman and Samuelson (1990) arises due to different interpretations of what a small punishment is: a proportional action in our case and a punishment propor-

[1] The equilibria analysis in the paper is erroneous, see Molinari (2000).

tional to gains in payoffs in their case. We discuss this crucial difference in more detail at the end of the paper. In their later work, Friedman and Samuelson (1994) showed that the Folk Theorem of Fudenberg and Maskin (1986) can be extended to games with continuous reaction functions.

Lastly, there is a group of papers that study prisoner's dilemma games with intermediate actions in an evolutionary setting: To (1988), Frean (1996), Wahl and Nowak (1999), Darwen and Yao (2002). All these papers assume either linear or restricted quadratic payoffs and none of them documents escalating dynamics. In contrast, we study whether there are payoff structures, not necessarily linear, that can lead to escalating dynamics.

2 General Analysis

In this section we present and discuss our general result. Consider an arbitrary prisoner's dilemma game:

$$\Gamma = \begin{array}{c|cc} A\backslash B & C & D \\ \hline C & R,R & S,T \\ D & T,S & P,P \end{array}$$

where $T > R > P > S$ and $2R > T + S$.

If this game is played once, then the only Nash equilibrium is (D, D). If we consider a repeated version of this game, then (C, C) can be sustained in an equilibrium if and only if the discount factor $\delta \geq \frac{T-R}{T-P}$.

Game Γ has two levels of cooperation: full cooperation and full defection. Broadly speaking, we want to ask the following question: what happens with the cooperative equilibrium if we add intermediate levels of conflict to game Γ? To make this question precise, we need to define what we mean by a game with intermediate levels of conflict or cooperation; we also need to define the class of strategies that we plan to study.

For any $N > 2$ we define class $\mathcal{G}_N$ of games with N levels of cooperation as follows. Each element $\Gamma_N \in \mathcal{G}_N$ is a game between two players, A and B, where each player can choose an action $a \in \{1, \ldots, N\}$. Choosing $a = 1$ means full cooperation, choosing $a = N$ means full defection. For each action choice (a, b) the payoff for player A is $u_A(a, b)$ and for player B it is $u_B(a, b)$. We consider symmetric games, namely $u_A(a, b) = u_B(b, a) = u(a, b)$. We further impose that for any a, b, and c such that $1 \leq a < b \leq N$ and $1 \leq c \leq N$ the following restrictions hold for the payoff matrix:

1. $u(1,1) = R$, $u(N,1) = T$, $u(1,N) = S$, $u(N,N) = P$,

2. $u(b,c) > u(a,c)$, $u(c,b) < u(c,a)$,

3. $u(a,a) > u(b,b)$, and $2u(a,a) > u(a,b) + u(b,a)$.

Condition 1 (consistency) means that full cooperation and full conflict lead to the same outcomes as in the original game Γ. Condition 2 (monotonicity) guarantees that intermediate actions generate intermediate payoffs. Finally, condition 3 (prisoner's dilemma) means that every 2×2 principal submatrix of the payoff matrix can itself be viewed as a prisoner's dilemma game. We impose Condition 3 to avoid local changes in the strategic nature of the game when intermediate actions are added. For example, this condition helps us to exclude games where local escalation is mutually profitable. Note that the games in a given $\mathcal{G}_N$ are characterised by the same set of players and actions but differ in their payoff functions, which, however, must be compatible with conditions 1–3.

We consider an infinitely repeated game, where each stage game is some fixed $\Gamma_N \in \mathcal{G}_N$. We assume that both players discount their payoffs with the same discount factor δ.

Finally, for a given N we limit our attention to equilibria in the class of strategies Σ_N, where each element $\sigma \in \Sigma_N$ is defined as follows:

1. start play with some action $a_0 \in \{1, \ldots, N-1\}$,

2. in any round t play $a_t = \max\{a_{t-1}, b_{t-1}\}$, where a_{t-1}, b_{t-1} are actions played in the previous round.

In other words, in Σ_N the punishment never exceeds the deviation. We therefore call such strategies "Markovian strategies with limited punishment."

Note that elements in Σ_N differ only in their starting points, namely action a_0. Further, as every $\Gamma_N \in \mathcal{G}_N$ has the same set of players and actions, Σ_N is well-defined for any $\Gamma_N \in \mathcal{G}_N$.

Any original game Γ has an equilibrium in Σ_2 strategies if δ is large enough. (We label C as 1 and D as 2.) This is the equilibrium where each player starts with $a = 1$, and $(1,1)$ remains a steady state from there on. However, as the number of actions increases, strategies with limited punishment might fail to deliver an equilibrium in Γ_N. This is formally captured by the following proposition.

Proposition 1. *For any $\delta < 1$ there is N large enough and a game $\Gamma_N \in \mathcal{G}_N$ such that no pair of strategies (σ, σ), with $\sigma \in \Sigma_N$, constitutes an equilibrium in Γ_N.*

Proof. See the appendix. □

Our proof is constructive. We explicitly build games Γ_N using a discretization of a suitably chosen continuous payoff function. Our construction is by no means unique and many other examples that lead to full conflict can be made. The main requirement for any such construction is that there are sufficient incentives for deviation around cooperative outcomes, while the payoffs remain sufficiently bounded so as to satisfy conditions 1–3. Such construction implies a sufficiently steep increase of the payoffs one step off the main diagonal, which, as we discuss later, can be characterized by the Lipschitz constant.

However, it is easy to see that not every game with a large number of intermediate actions leads to a break-down of cooperation. For example, if the game is extended "uniformly," i.e. with payoffs defined linearly along the main diagonal, linearly above it (triangle $R-S-P$), and linearly below it (triangle $R-T-P$), then all the incentives of the original game are preserved, the critical value of the discount factor remains the same, and the additional intermediate actions do not result in escalation.[2]

A generalization of "uniformly" extended games where cooperation would survive the addition of intermediate actions are games with sufficiently smooth payoffs. In what follows we provide a partial characterization of this class of games.

Define

$$\tilde{\mathcal{G}}_N = \{\Gamma_N \in \mathcal{G}_N : u(a,a) = R - (R-P)(a-1)/(N-1)\ \forall\, a\}.$$

That is, $\tilde{\mathcal{G}}_N$ is a restriction of $\mathcal{G}_N$ to games with payoffs that are uniformly spaced along the main diagonal. Our example in the introduction belongs to this class of games. Further, for $K \in \mathbb{R}_{>0}$ define

$$\hat{\mathcal{G}}_N(K) = \left\{\Gamma_N \in \mathcal{G}_N : |u(a,b) - u(c,d)| \leq \frac{K}{N-1} \max\{|a-c|, |b-d|\}\ \forall\, a,b,c,d\right\}.$$

Namely, $\hat{\mathcal{G}}_N(K)$ is a restriction of $\mathcal{G}_N$ to games with an upper bound on how fast payoffs can change in response to changes in actions. Formally, $\hat{\mathcal{G}}_N(K)$ is a restriction of $\mathcal{G}_N$ to games with Lipschitz-continuous payoffs of modulus K, where for any given N the action space metric that we use is the maximum metric scaled by $N-1$. Finally, denote

$$\delta_0 = 1 - \frac{R-P}{2(T-S)}.$$

With these definitions in mind, we have

[2] It is straightforward to show that this "uniform" extension belongs to class $\mathcal{G}_N$.

Proposition 2. *For any $\delta \geq \delta_0$, $N \geq 2$ and for any $K \in \left[\frac{1}{1-\delta_0}(R-P), \frac{1}{1-\delta}(R-P)\right]$, $\tilde{\mathcal{G}}_N \cap \hat{\mathcal{G}}_N(K)$ is non-empty and for any game $\Gamma_N \in \tilde{\mathcal{G}}_N \cap \hat{\mathcal{G}}_N(K)$ any pair of strategies (σ, σ), with $\sigma \in \Sigma_N$, constitutes an equilibrium in Γ_N.*

Proof. See the appendix. □

Proposition 2 gives sufficient, not necessary, conditions for the survival of cooperation in games with intermediate actions. Other classes of games where cooperation survives exist. For example, if there is a big "gap" in the payoffs somewhere along the main diagonal of the game, the conflict escalation naturally stops at that gap, provided that the discount factor is sufficiently large.

Figure 1 illustrates Propositions 1 and 2 as applied to the base game Γ from the introduction. The shaded area in the left panel gives the values of (N, δ) used in the construction of Proposition 1, see equation (5) in the appendix, and for which there are provably games Γ_N that admit no equilibria in Markovian strategies with limited punishment. The upper boundary of the shaded area approaches $\delta = 1$ as $N \to \infty$. The shaded area in the right panel corresponds to the statement of Proposition 2 and gives those values of (K, δ) for which, for any $N \geq 2$ any game Γ_N provably admits an equilibrium in Markovian strategies with limited punishment.

As has been noted earlier, Propositions 1 and 2 give sufficient but not necessary conditions regarding the non-existence or existence of games with sustainable cooperation. In other words, the propositions do not give tight bounds on the corresponding regions in (N, K, δ). However, for a specific base game, we can compute those bounds numerically in a straightforward fashion. Let

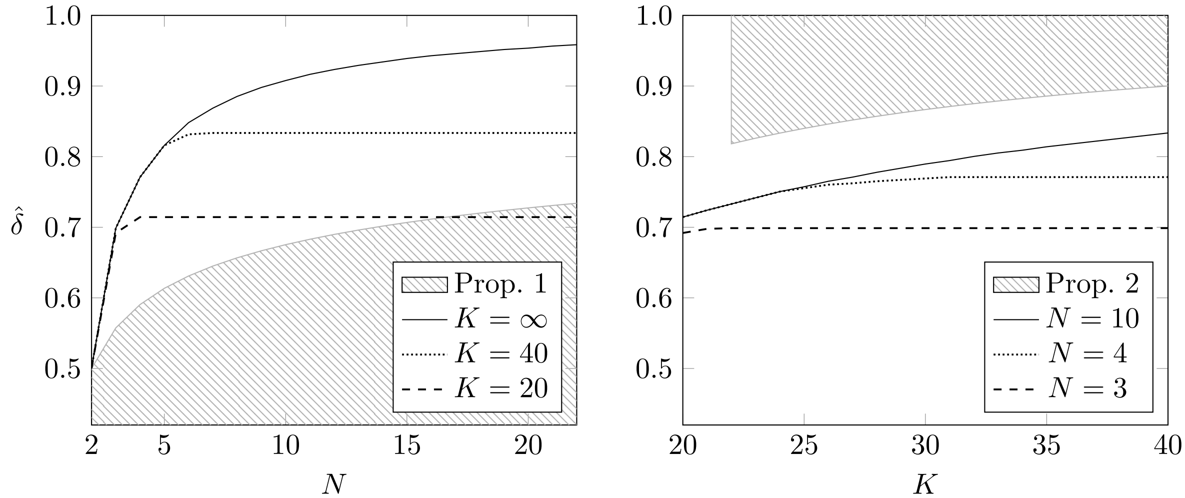
$$E_N(\delta) = \left\{\Gamma_N \in \mathcal{G}_N : u(a+1, a) + \frac{\delta}{1-\delta} u(a+1, a+1) > \frac{1}{1-\delta} u(a, a) \; \forall\, a : 1 \leq a < N\right\}$$

be a set of games Γ_N with profitable one-stage deviations along the main diagonal. For all K such that $\tilde{\mathcal{G}}_N(K)$ is not empty we define

$$\hat{\delta}(N, K) = \sup\left\{\delta : \tilde{\mathcal{G}}_N(K) \cap E_N(\delta) \neq \emptyset\right\}.$$

Then for any $\delta < \hat{\delta}$ there exist games $\Gamma_N \in \tilde{\mathcal{G}}_N(K) \subseteq \mathcal{G}_N$ that do not have equilibria in Σ_N. And conversely, for any $\delta > \hat{\delta}$ any game $\Gamma_N \in \tilde{\mathcal{G}}_N(K)$ admits an equilibrium in Σ_N (albeit, there might be just a single place along the main diagonal where escalation is prevented).

Figure 1: Maximum δ Where Cooperation Can Break Down



We use $S = 0$, $P = 3$, $R = 7$, $T = 11$. The shaded area in the left panel shows the values of (N, δ) used in the proof of Proposition 1. The shaded area in the right panel shows the values of (K, δ), where Proposition 2 is applicable. The solid, dotted, and dashed lines show $\hat{\delta}(N, K)$, i.e. the maximum discount factor for which games can be constructed where cooperation breaks down.

For a given (N, K, δ), sets $\mathcal{G}_N$, $\tilde{\mathcal{G}}_N(K)$ and $E_N(\delta)$ are all defined by linear inequalities. Therefore a simplex method[3] can be used to efficiently check whether $\tilde{\mathcal{G}}_N(K) \cap E_N(\delta) \neq \emptyset$. Given that $\mathcal{I}(\tilde{\mathcal{G}}_N(K) \cap E_N(\delta) \neq \emptyset)$ is clearly monotone in δ, we determine $\hat{\delta}$ using the bisection method, starting with the endpoints $\delta = 0$ and $\delta = 1$.

Figure 1 shows $\hat{\delta}(N, K)$ as a function of N for various K as well as a function of K for various N. In the left panel, the shaded area lies below $\hat{\delta}(N, \infty)$. That is, our numerical analysis shows that the set of values (N, δ) for which there are games where cooperation breaks down is strictly larger than the set we consider in the proof of Proposition 1. That is, Proposition 1 gives sufficient but not necessary conditions. An analogous reasoning applies to Proposition 2 as the shaded area in the right panel lies strictly above $\hat{\delta}(N, K)$ for any N, i.e. there are values of discount factor lower than the ones considered in the proposition, for which cooperation can be sustained in all games with intermediate actions.

We do not focus on the precise mechanics of escalation when the players find themselves in a cooperative outcome that cannot be supported as an equilibrium, i.e. if no equilibrium exists in strategies from the class Σ_N. It is conceivable that similarly to institutional re-

[3]For computing Figure 1 we use the simplex method from NumPy, which in turn uses HiGHS (Huangfu and Hall, 2018).

strictions on large punishments there might be institutional restrictions on large deviations. In such cases the conflict would escalate in a sequence of small deviations and gradually approach the most non-cooperative outcome.

3 Discussion

Our result is robust to the assumption that punishment must be exactly symmetric. Any moderate degree of asymmetry, in a sense that a deviation to action a can be punished by at most action $a+k$, where k is a fixed number, still leads to the same conclusion, but possibly requires a higher N. It is critical, however, that the loss in the payoff from the strongest feasible punishment is not too high. This can be guaranteed by a construction algorithm similar to the one we use in our proof as long as k does not depend on N.

We have assumed that every escalation step takes the same fixed time. If smaller deviations become faster with the addition of new actions, then whether there is full escalation or not depends on the rate of the speed increase and on the convexity of the initial payoffs, but full escalation can emerge even in this case.

Our finding contrasts to that in Schelling (1980), who argued that intermediate levels of conflict and punishment make cooperation more stable. His logic was based on the idea that small threats are more credible and therefore act a sufficient deterrence device. Our analysis suggests that, although being credible, these small punishments might not be sufficiently grim, which results in step-by-step escalation of conflict.

The relation between our paper and Friedman and Samuelson (1990) is of particular interest. Friedman and Samuelson show that cooperative outcomes can be achieved in games with continuous strategies, where small deviations are met with small punishments. We consider similar strategy profiles but we focus on discrete rather than continuous games, and we show that in discrete games cooperation can break down. Let us elaborate on these seemingly contradictory conclusions.

The crucial difference between our papers lies in how we define "small punishments". In Friedman and Samuelson (1990), a punishment is considered "small" if it is proportional to the gains of a deviator, while in our case a punishment is considered "small" if it is proportional to the deviation distance in the action space. If we were to introduce Friedman and Samuelson's concept into our discrete games, then punishments could be strong enough to prevent any escalation. In particular, our Proposition 1 would not hold. Similarly, if our concept of small punishments was introduced into Friedman and Samuelson, then full escalation might happen for some choices of the payoff function.[4] Thus, our concept of a small, proportional punishment makes our results differ both from the results found in the usual repeated games literature and from those by Friedman and Samuelson.

Appendix

Proof of Proposition 1. We prove the proposition as follows. First, we choose a continuous payoff function such that any uniform discretization of this function satisfies conditions 1, 2, and 3 (consistency, monotonicity, and prisoner's dilemma). Second, we show that for any $\delta < 1$ there is a discretization that is sufficiently fine so that the corresponding game does not have an equilibrium in Σ_N.

Choose L to be the smallest integer but no less than 3 such that

$$L > \begin{cases} \dfrac{T-R}{P-S} + 1 & \text{if} \quad \dfrac{T-R}{R-P} \leq 1, \\ \dfrac{T-R}{T-2R+P} + 1 & \text{otherwise.} \end{cases}$$

Note that $L \geq 3$. For $a \in \{1, \ldots, N\}$ and $b \in \{1, \ldots, N\}$ let

$$u(a,b) = f\left(\frac{a-1}{N-1}, \frac{b-1}{N-1}\right), \tag{1}$$

where

$$f(x,y) = \begin{cases} (T-R)(x-y)^{1-1/L} + R - (R-P)y & \text{if} \quad x \geq y, \\ (S-P)(y-x)^{1/L} + R - (R-P)y & \text{if} \quad x < y, \end{cases} \tag{2}$$

is a continuous function defined on $[0,1] \times [0,1]$. Note that L and f do not depend on N.

We verify conditions 1, 2, and 3 now. We have

$$u(1,1) = f(0,0) = R, \quad u(N,1) = f(1,0) = T,$$
$$u(1,N) = f(0,1) = S, \quad u(N,N) = f(1,1) = P.$$

[4] Our payoff function (2) is one such example. Note that it is not Lipschitz-continuous along the main diagonal of the action space and therefore violates the conditions of Theorem 1 in Friedman and Samuelson (1990).

So, Condition 1 is satisfied. Next, we have

$$\frac{\partial}{\partial x} f(x,y) = \begin{cases} \dfrac{L-1}{L}(T-R)(x-y)^{-1/L} > 0 & \text{if} \quad x > y, \\ \dfrac{1}{L}(P-S)(y-x)^{1/L-1} > 0 & \text{if} \quad x < y, \end{cases}$$

$$\frac{\partial}{\partial y} f(x,y) = \begin{cases} -\dfrac{L-1}{L}(T-R)(x-y)^{-1/L} - (R-P) < 0 & \text{if} \quad x > y, \\ -\dfrac{1}{L}(P-S)(y-x)^{1/L-1} - (R-P) < 0 & \text{if} \quad x < y. \end{cases}$$

Therefore $f(x,y)$ is strictly increasing in x and strictly decreasing in y. Consequently, Condition 2 is satisfied.

Note that

$$\frac{\partial}{\partial x} f(x,x) = -(R-P) < 0,$$

hence f is strictly decreasing along its main diagonal and the first part of Condition 3 is satisfied. The second part of the condition requires that $2u(a,a) > u(a,b) + u(b,a)$ for any integer a, b such that $1 \le a < b \le N$. Using our definition of u and rearranging terms, we obtain

$$(R-P)\frac{b-a}{N-1} - (T-R)\left(\frac{b-a}{N-1}\right)^{1-1/L} + (P-S)\left(\frac{b-a}{N-1}\right)^{1/L} > 0$$

or, equivalently,

$$(R-P)\left(\frac{b-a}{N-1}\right)^{1-1/L} - (T-R)\left(\frac{b-a}{N-1}\right)^{1-2/L} + (P-S) > 0. \tag{3}$$

Let

$$g(\phi) = (R-P)\phi^{1-1/L} - (T-R)\phi^{1-2/L} + (P-S).$$

Then to show that (3) holds for any integer a, b such that $1 \le a < b \le N$ it suffices to show that $g(\phi) > 0$ for all $\phi \in [0,1]$.

Given that $L \ge 3$, we have that $g(\phi)$ is continuous and bounded on $[0,1]$. Hence, we only need to check the sign of g at its boundary and inflection points. We have $g(0) = P - S > 0$ and $g(1) = 2R - (T+S) > 0$ (convexity of the original game).

Solving $g'(\phi) = 0$ we obtain that g has a unique inflection point on $(0,\infty)$ given by

$$\phi_0 = \left(\frac{T-R}{R-P}\,\frac{L-2}{L-1}\right)^L.$$

Suppose that $\frac{T-R}{R-P} > 1$. We have required that $L > \frac{T-R}{T-2R+P} + 1$. From the first inequality if follows that $T - 2R + P > 0$, and therefore the second inequality yields $\frac{L-2}{L-1} > \frac{R-P}{T-R}$.

Consequently, $\phi_0 > 1$. There are thus no inflection points on $[0,1]$, and so $g(\phi) > 0$ for all $\phi \in [0,1]$.

Suppose that $\frac{T-R}{R-P} \le 1$. Then $\phi_0 < 1$. Evaluating g at ϕ_0 and rearranging, we get

$$g(\phi_0) = (P-S) - \frac{T-R}{L-1}\phi_0^{1-2/L} > (P-S) - \frac{T-R}{L-1}.$$

We have required that $L > \frac{T-R}{P-S} + 1$. It immediately follows that $g(\phi_0) > 0$. So, g is strictly positive at its boundary points as well as at its unique interior inflection point. Hence, $g(\phi) > 0$ for all $\phi \in [0,1]$. Summarizing, we shown that the second part of Condition 3 holds.

Having payoffs u as defined in (1), we proceed to show that given any $\delta < 1$ there exists a sufficiently large N so that no symmetric pair of strategies from Σ_N forms an equilibrium.

Consider a pair of strategies (σ, σ), with $\sigma \in \Sigma_N$. A necessary condition for these strategies to form an equilibrium is that the first player does not have an incentive to deviate from some steady state (a,a) to $(a+1,a)$, or

$$\frac{1}{1-\delta}u(a,a) \ge u(a+1,a) + \frac{\delta}{1-\delta}u(a+1,a+1). \tag{4}$$

Conversely, if this condition is not satisfied, then no such pair of strategies forms an equilibrium. Expanding u in (4) and rearranging terms we obtain

$$\frac{\delta}{1-\delta}\,\frac{R-P}{N-1} \ge \frac{T-R}{(N-1)^{1-1/L}}$$

and therefore for

$$N > \left(\frac{\delta}{1-\delta}\,\frac{R-P}{T-R}\right)^L + 1 \tag{5}$$

the equilibrium does not exist. Thus, given any $\delta < 1$ we can choose N sufficiently large so that (4) does not hold. For such an N, no pair of strategies (σ, σ), with $\sigma \in \Sigma_N$, constitutes an equilibrium. □

Proof of Proposition 2. Firstly, let us verify that $\tilde{\mathcal{G}}_N \cap \hat{\mathcal{G}}_N(K)$ is non-empty. Consider

$$u(a,b) = \begin{cases} R + (T-R)\dfrac{a-1}{N-1} - (T-P)\dfrac{b-1}{N-1} & \text{if} \quad a \ge b, \\ R + (P-S)\dfrac{a-1}{N-1} - (R-S)\dfrac{b-1}{N-1} & \text{if} \quad a < b. \end{cases}$$

Function $u(a,b)$ gives piece-wise linear payoffs which "bend" around the main diagonal. It is straightforward to verify that consistency, monotonicity, and prisoner's dilemma conditions are satisfied, thus $u(a,b)$ defines a game $\Gamma_N \in \mathcal{G}_N$. We have

$$u(a,a) = R - (R-P)\frac{a-1}{N-1},$$

therefore $\Gamma_N \in \tilde{\mathcal{G}}_N$.

Now, consider any a,b,c,d such that $a \geq b$ and $c \geq d$. We have

$$|u(a,b) - u(c,d)| = \left|(T-R)\frac{a-c}{N-1} - (T-P)\frac{b-d}{N-1}\right| \leq \frac{T-R}{N-1}|a-c| + \frac{T-P}{N-1}|b-d| \leq 2\frac{T-S}{N-1}\max\{|a-c|,|b-d|\} \leq \frac{K}{N-1}\max\{|a-c|,|b-d|\}$$

whenever

$$K \geq 2(T-S) = \frac{1}{1-\delta_0}(R-P).$$

Analogous inequality holds for any a,b,c,d where $a \leq b$ and $c \leq d$.

Now, consider any a,b,c,d such that $a \geq b$ and $c < d$. If we draw a line from (a,b) to (c,d), this line will intersect the main diagonal at (m,m), where $m = \frac{ad-bc}{a+d-b-c}$. Point m need not be an integer, i.e. it need not belong to the action space of game Γ_N. In this case, for any $K \geq \frac{1}{1-\delta_0}(R-P)$ we have

$$|u(a,b) - u(c,d)| \leq |u(a,b) - u(m,m)| + |u(m,m) - u(c,d)| \leq \frac{K}{N-1}\max(|a-m|,|b-m|) + \frac{K}{N-1}\max(|m-c|,|m-d|) = \frac{K}{N-1}\max(|a-c|,|b-d|),$$

where the last equality holds, because the maximum norm is additive for segments of a line. Analogous inequality holds when $a < b$ and $c \geq d$. Summarizing, $\Gamma_N \in \hat{\mathcal{G}}_N(K)$.

Secondly, let us show that any pair of strategies (σ,σ), with $\sigma \in \Sigma_N$, makes an equilibrium in Γ_N. A necessary and sufficient condition for that to be the case is that no profitable one-stage deviations exist. Namely, for any tuple of actions (a,b), with $b > a$, we require

$$u(a,a) \geq (1-\delta)u(b,a) + \delta u(b,b)$$

or, equivalently,

$$u(a,a) - u(b,b) \geq (1-\delta)\big(u(b,a) - u(b,b)\big). \tag{6}$$

Now, as $\Gamma_N \in \tilde{\mathcal{G}}_N$ we have $u(a,a) - u(b,b) = \frac{R-P}{N-1}(b-a)$. Moreover, as $\Gamma_N \in \hat{\mathcal{G}}_N$ we have $u(b,a) - u(b,b) \leq \frac{K}{N-1}(b-a)$. Thus, (6) holds if

$$\frac{R-P}{N-1}(b-a) \geq (1-\delta)\frac{K}{N-1}(b-a),$$

which is satisfied for all $K \leq (R-P)/(1-\delta)$. □

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